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Slip flow past a tangential flat plate at low Reynolds numbers

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In this paper the slip flow of viscous fluid at low Reynolds numbers past a flat plate aligned with the flow is studied theoretically on the basis of Oseen–Stokes equations of motion. An integral equation for the distribution of fundamental singularities representing the plate is derived and solved approximately in the vicinity of the edge and main portion of the plate. A formula for the local skin friction is obtained and discussed numerically. It is also shown that the slippage of the flow gives rise to reduction of the drag force on the plate by an amount $O(K|\ln K|)$, where K is the Knudsen number. The velocity change near the edge of the plate is of particular interest and is found to be logarithmically singular there.

1. Introduction

It is well known that the flow of a rarefied gas past a solid body exhibits slipping at the body surface. If the rarefaction is slight, the (macroscopic) slip velocity is proportional to the shear stress at the surface according to kinetic theory. The constant of proportionality is of the order of the mean free path, so that the slip velocity is usually small. Thus most work on the slip flow has been restricted to the study of small perturbations about no-slip solutions (Donaldson 1949; Schaaf & Chambré 1958). The effect of slip is expected however to be large at the leading edge of a flat plate since the shear stress becomes infinite there if the flow does not slip. Flows of this sort cannot be studied by simple perturbation analysis. One approach may be to use the slip boundary condition with no restriction on the magnitude of the slip velocity (though there is no rigorous support from kinetic theory at present). Laurmann (1961) treated, along these lines, incompressible slip flow past a semi-infinite flat plate at zero incidence on the basis of Oseen's linearized equation of motion. He derived, among others, formulae for the local skin-friction coefficient c_f at points near the leading edge and far downstream. He noticed that boundary-layer theory using the slip condition does not predict the nature of the solution at small Mach numbers (or small Reynolds numbers based on the mean free path) correctly, even at the surface of the plate. Laurmann also obtained a formula giving c_f at any point on the plate for the case of flow at low Reynolds numbers as a particular limit.

Unfortunately there seem to be some errors in this formula and related results. The case of low-speed flow is of importance since it is in this regime that conclusive comparisons of theory and experiment can be made. Therefore the present paper aims to reconsider the case of low Reynolds number flow directly by a different approach from Laurmann's. The plate is here supposed to be of finite length. An integral equation is derived in § 2 for the distribution of fundamental solutions of the Oseen–Stokes equation, which represents the effect of the plate on otherwise uniform flow under the slip condition. In § 3 an approximate solution valid in the vicinity of the edge of the plate is found by means of the Wiener–Hopf technique. A formula for c_f is given and discussed numerically. In § 4 a perturbation solution for the central portion of the plate is also obtained and a uniformly valid solution is constructed by a matching procedure. Then the drag force on the plate is calculated and it is shown that the slipping of the flow reduces the drag by an amount $O(K|\ln K|)$, where K is the Knudsen number. Approximate expressions for the velocity field near the leading edge of the plate are also obtained, revealing a special type singularity there. The corresponding result for the trailing edge can be found at once by virtue of flow symmetry.

2. Basic equations

We consider two-dimensional steady flow of a viscous fluid with uniform velocity U at infinity past a flat plate of length l aligned with the flow. The flow is supposed to obey the Oseen linearized equation of motion and undergo slipping at the plate surface with velocity proportional to the shear stress there. We take Cartesian coordinates (x, y) normalized by the plate length l in the plane of fluid motion. The x axis is taken parallel to the direction of the uniform flow and the plate lies on the x axis between $x = 0$ and $x = l$. Let $1+u$ and v be the x and y components of the velocity at any point normalized by U. Then the equation of continuity and the Oseen equation of motion may be written as

$$\partial u/\partial x + \partial v/\partial y = 0, \tag{1}$$

$$\frac{\partial \omega}{\partial x} = \frac{1}{R}\Delta\omega, \quad \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \tag{2}$$

where ω is the vorticity, $R = lU/\nu$ the Reynolds number, ν the kinematic coefficient of viscosity and $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$. A fundamental solution of these equations which represents the asymptotic field far from an obstacle, an Oseenlet (Rosenhead 1963), is known to be

$$u - iv = \bar{C}(\tfrac{1}{2}Rre^{i\theta})^{-1} - \exp(\tfrac{1}{2}Rx)\{CK_0(\tfrac{1}{2}Rr) + \bar{C}K_1(\tfrac{1}{2}Rr)\,e^{-i\theta}\}, \tag{3}$$

where $x + iy = re^{i\theta}$, K_0 and K_1 are modified Bessel functions and C is an arbitrary constant, $\bar{C}$ being the conjugate complex of C. We can express the perturbation velocity field in our problem as a suitable distribution of singularities of this type along the flat plate as follows:

$$u - iv = \frac{1}{2\pi}\int_0^1 \left[\frac{2}{R\tilde{r}}e^{-i\tilde{\theta}} - \exp(\tfrac{1}{2}R\tilde{x})\{K_0(\tfrac{1}{2}R\tilde{r}) + K_1(\tfrac{1}{2}R\tilde{r})\,e^{-i\tilde{\theta}}\}\right] f(\xi)\,d\xi, \tag{4}$$

where $\tilde{x} = x - \xi$, $\tilde{r} = (\tilde{x}^2 + y^2)^{\frac{1}{2}}$ and $\tilde{\theta} = \tan^{-1}(y/\tilde{x})$.

On the other hand, the slip boundary condition at the surface of the plate is given by kinetic theory in the form:

$$1+u = k|\partial u/\partial y| \quad (y = 0, \quad 0 < x < 1), \tag{5}$$

where the slip coefficient k is a small parameter proportional to the Knudsen number K, the ratio of the mean free path of a gas molecule to the length of the plate. Smallness of the right-hand side of (5) is assumed in the kinetic theory. However, in the first half of the present analysis we use condition (5) with no restriction on the magnitude of the slip velocity as mentioned earlier. Now, it follows from (4) that

$$\partial u/\partial y = \pm f(x) \quad (y = \pm 0, \quad 0 < x < 1). \tag{6}$$

Introducing (4) and (6) in (5), we obtain an integral equation for the distribution function $f(x)$ of the form

$$1+\frac{1}{2\pi}\int_0^1\left[\frac{2}{R(x-\xi)} - e^{\frac{1}{2}R(x-\xi)}\{K_0(\tfrac{1}{2}R|x-\xi|) + \operatorname{sgn}(x-\xi)\,K_1(\tfrac{1}{2}R|x-\xi|)\}\right]f(\xi)\,d\xi = kf(x) \quad (0 < x < 1). \tag{7}$$

In the case of interest here, i.e. when the Reynolds number R is much smaller than unity, the integral equation can be simplified by the kernel approximation:

$$1+\frac{1}{2\pi}\int_0^1\left(\ln|x-\xi| - \ln\frac{4}{R}+\gamma-1\right)f(\xi)\,d\xi = kf(x) \quad (0 < x < 1), \tag{8}$$

where $\gamma = 0{\cdot}5772\ldots$ is Euler's constant. This replacement of the kernel corresponds to application of the Stokes approximation to the fluid motion. In fact, convective effects can be safely neglected for the flow near the plate at small Reynolds numbers.

The simplified equation (8) is, however, not yet amenable to straightforward procedures. We shall try first to find an approximate solution valid near the edge of the plate, where the slipping of fluid at the surface becomes conspicuous as remarked before. Differentiating (8) with respect to x, we obtain

$$\frac{1}{2\pi}\int_0^1\frac{f(\xi)}{x-\xi}\,d\xi = k\frac{df}{dx} \quad (0 < x < 1), \tag{9}$$

where the Cauchy principal value of the integral is taken. In order to get the leading-edge solution, we use stretched variables defined as

$$t = k^{-1}x, \quad \tau = k^{-1}\xi. \tag{10}$$

Then, since k^{-1} is very large, (9) may be approximated as follows:†

$$\int_0^\infty \frac{g(\tau)}{t-\tau}\,d\tau - 2\pi\frac{dg}{dt} = 0 \quad (t > 0), \tag{11}$$

where

$$g(t) = f(kt). \tag{12}$$

† It is anticipated from the analysis of no-slip flow that the function $f(x)$ behaves like $x^{-\frac{1}{2}}$ for $1 \gg x \gg k$. If the co-ordinate stretching (10) is applied to (8) without differentiation, the replacement of k^{-1} by infinity makes the integral divergent and therefore becomes meaningless.

3. Edge solution

The integral equation (11) can be solved by means of the Wiener–Hopf technique. We first rewrite (11) in a form such that Fourier transformation may be applied:†

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} \epsilon K_1(\epsilon|t-\tau|)\,\mathrm{sgn}\,(t-\tau)\,g_-(\tau)\,d\tau - 2\pi\frac{dg_-}{dt} = h_+(t), \tag{13}$$

where

$$g_-(t) = \begin{cases} g(t) & (t>0), \\ 0 & (t<0), \end{cases} \qquad h_+(t) = \begin{cases} 0 & (t>0), \quad (14a) \\ h(t) & (t<0), \quad (14b) \end{cases}$$

and $h(t)$ is a function as yet unknown. By making use of the convolution theorem, we obtain the Fourier transform of (13) in the form

$$2\pi i p\Lambda(p)\,G_-(p) = -H_+(p), \tag{15}$$

where

$$G_-(p) = \int_{-\infty}^{\infty} g_-(t)\,e^{-ipt}\,dt, \tag{16a}$$

$$H_+(p) = \int_{-\infty}^{\infty} h_+(t)\,e^{-ipt}\,dt, \tag{16b}$$

$$\Lambda(p) = \lim_{\epsilon\to 0}\{1+\tfrac{1}{2}(p^2+\epsilon^2)^{-\frac{1}{2}}\} \tag{16c}$$

and the subscripts $\pm$ indicate regularity of the function in the upper and lower halves of the p plane respectively.

In order to find $G_-(p)$ from (15), it is necessary to factorize the function $\Lambda(p)$ in the form

$$\Lambda(p) = \Lambda_-(p)/\Lambda_+(p). \tag{17}$$

This can be done by a standard procedure using the Cauchy integral representation for $\ln\Lambda(p)$ with a deformed contour of integration around the real axis; we then have

$$\ln\Lambda_-(p) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{1}{q-p}\ln\Lambda(q)\,dq, \tag{18}$$

where the path of integration avoids the point $q=p$ via a small semicircle traversed in the clockwise sense. The integral may be simplified: on differentiation with respect to p and integration by parts, we get

$$\begin{aligned}\frac{1}{\Lambda_-}\frac{d\Lambda_-}{dp} &= -\frac{1}{2p(1+2|p|)}+\frac{1}{\pi i}\int_0^{\infty}\frac{1}{(q^2-p^2)(1+2q)}dq \\ &= -\frac{1}{2p(1+2|p|)}+\frac{2i}{\pi(4p^2-1)}\ln 2|p|. \end{aligned}\tag{19}$$

Further integration gives

$$\ln\Lambda_-(p) = \tfrac{1}{2}\ln\left(1+\frac{1}{2|p|}\right)+\frac{i}{2\pi}\mathrm{sgn}\,(p)\{P(|p|)-\tfrac{1}{2}\pi^2\}, \tag{20}$$

where the constant of integration has been so chosen that $\ln\Lambda_-(p)$ tends to zero as $|p|$ becomes infinite, and

$$P(p) = \int_0^p \frac{\ln(2s)}{s^2-\frac{1}{4}}ds, \tag{21}$$

† We replace the kernel of (11) as in (13) in order to obtain the existence of the Fourier transform. After the nature of the solution has been obtained, we can let $\epsilon\to 0$.

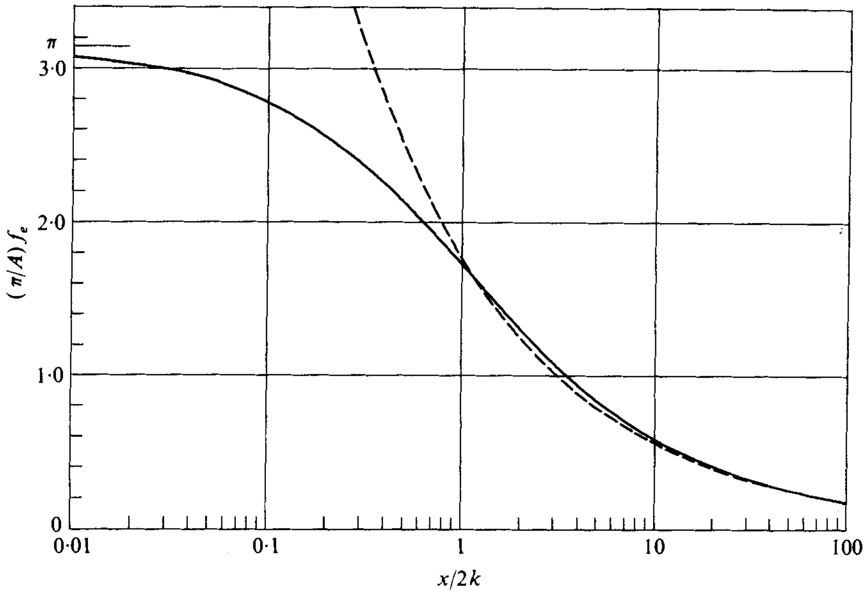

FIGURE 1. Distribution of shearing stress in the edge region. ——, present result; – – –, no-slip flow.

in which $\ln 2s$ is taken to be real for positive s. Below, $P(p)$ is defined for arbitrary p by analytic continuation. Thus $\Lambda_-(p)$ is given for real p by

$$\Lambda_-(p) = \left(1+\frac{1}{2|p|}\right)^{\frac{1}{2}} \exp\left[\frac{i}{2\pi}\operatorname{sgn}(p)\{P(|p|)-\tfrac{1}{8}\pi^2\}\right]. \tag{22}$$

Now, when (15) is rewritten in the form

$$ip\Lambda_-(p)\,G_-(p) = -\Lambda_+(p)\,H_+(p)/2\pi, \tag{23}$$

we see that the left- and right-hand sides are analytic in the lower and upper halves of the p plane respectively. Therefore, both sides represent a single entire function which is shown to be a constant A (say) to ensure the existence of Fourier inversion. Thus the function $G_-(p)$ is found to be

$$G_-(p) = -iA/p\Lambda_-(p), \tag{24}$$

where $\Lambda_-(p)$ is given by (22). Fourier inversion of (24) together with (10) and (12) gives the required edge solution $f_e(x)$ of (8) in the form

$$f_e(x) = \frac{A}{\pi}\int_0^\infty \frac{1}{[p(p+\frac{1}{2})]^{\frac{1}{2}}}\sin\left\{\frac{x}{k}p-\frac{1}{2\pi}P(p)+\frac{\pi}{4}\right\}dp. \tag{25}$$

This edge solution is valid in the neighbourhood of the leading edge of the plate ($x \ll 1$). The scale factor A is to be determined by the process of matching with the solution valid in the central part of the plate (Van Dyke 1964). This matching will be considered in the next section. Figure 1 shows $\pi f_e(x)/A$, which is proportional, according to (6), to the local shearing stress at the surface. It will be seen that the shearing stress remains finite at the leading edge and becomes smaller downstream. The asymptotic behaviour of the edge solution is as follows:†

$$f_e(x) \sim \begin{cases} A\left\{1+\dfrac{x}{2\pi k}\ln\dfrac{x}{k}+(\gamma-\ln 2-2)\dfrac{x}{2\pi k}+\ldots\right\} & \text{for} \quad x \ll k, \qquad (26) \\ A\left(\dfrac{2k}{\pi x}\right)^{\frac{1}{2}}\left\{1+\dfrac{k}{\pi x}\ln\dfrac{x}{k}+(\gamma+\ln 2-1)\dfrac{k}{\pi x}+\ldots\right\} & \text{for} \quad x \gg k. \qquad (27) \end{cases}$$

Laurmann (1961) treated incompressible slip flow past a semi-infinite flat plate at zero incidence on the basis of the Oseen equation. He obtained a formula for the local shear stress corresponding to (25) as the small Reynolds number limit of a general case. However, his result does not agree with ours. Therefore we have followed through the analysis leading to the above-mentioned limit (Laurmann did not give the details of his calculation). Thus, starting from the general results (4.7) and (4.11) of Laurmann, introducing $t = x/a_1\lambda$ [which is just our $t = x/k$ in (10)] and making the transformation $W = 2p/\Lambda$ (in Laurmann's notation), we took the limit $\Lambda \to 0$. This process does not lead to Laurmann's equation (5.7) but yields the present equation (25) except for a constant. The constant factors may differ in the cases of a finite and a semi-infinite plate respectively. It would be unfortunate if some errors occurred in Laurmann's detailed calculations. One sees by comparing Laurmann's figure 4 with our figure 1 that his formula predicts values of the shear stress which are too large (by 50 % or more) at and near $x/2k = 1$. Also, the related result shown in his figure 5 seems to need correction.

4. Uniformly valid solution and drag formula

The no-slip boundary condition holds approximately over most of the plate except for narrow edge regions. Therefore the main solution $f_m(x)$ to the first approximation satisfies the integral equation (8) with the right-hand side set to zero. The no-slip problem for the flat plate has been already treated by Piercy & Winny (1933) and the solution is

$$f_m(x) = \frac{2}{\ln(16/R)-\gamma+1}\frac{1}{[x(1-x)]^{\frac{1}{2}}}. \qquad (28)$$

The scale factor A in the edge solution (25) can be determined by matching it with the main solution in the region where $1 \gg x \gg k$. Thus, comparing (27) with (28) for small x, we get

$$A = \frac{1}{\ln(16/R)-\gamma+1}\left(\frac{2\pi}{k}\right)^{\frac{1}{2}}. \qquad (29)$$

† To obtain the asymptotic formulae, it is convenient to rewrite (25) in the form

$$f_e(x) = \frac{A}{2\pi i}\int_{-\infty}^{\infty}\frac{1}{[p(p+\frac{1}{2})]^{\frac{1}{2}}}\exp\left\{\frac{ix}{k}p-\frac{i}{2\pi}P(p)+\frac{\pi}{4}i\right\}dp,$$

where the path of integration is deformed around the point $p = 0$ in the counter-clockwise sense. Note that $p = -\frac{1}{2}$ is not a singular point of the integrand. Evaluation of the contributions from large and small p to the integral yields (26) and (27) respectively (cf. Carslaw & Jaeger 1947, p. 271).

Next we consider the solution for the trailing-edge region. It is sufficient for this to note that, even under the slip boundary condition (5), the solution to the Stokes approximation is symmetric with respect to the straight line normal to the plate at its centre. Accordingly, the trailing-edge solution is also given by (25) but with its argument x replaced by $1-x$.

In terms of the edge and main solutions obtained above, we can construct a solution uniformly valid over the whole of the plate to the first approximation in the sense of additive composition (cf. Van Dyke 1964, p. 94):

$$f_u(x) = \frac{2}{\ln(16/R)-\gamma+1}\left[\frac{1}{[x(1-x)]^{\frac{1}{2}}} - \frac{1}{x^{\frac{1}{2}}} - \frac{1}{(1-x)^{\frac{1}{2}}} + \frac{1}{(2\pi k)^{\frac{1}{2}}}\int_0^\infty \frac{1}{[p(p+\frac{1}{2})]^{\frac{1}{2}}}\left\{\sin\left(\frac{x}{k}p - \frac{1}{2\pi}P(p)+\frac{\pi}{4}\right) + \sin\left(\frac{1-x}{k}p - \frac{1}{2\pi}P(p)+\frac{\pi}{4}\right)\right\}dp\right]. \quad (30)$$

Calculation of the drag experienced by the plate requires integration of $f(x)$ over its surface. By integrating (30), we get (see appendix A)

$$\int_0^1 f_u(x)\,dx \doteqdot \frac{2}{\ln(16/R)-\gamma+1}\left\{\pi - \frac{4k}{\pi}\left(\ln\frac{2}{k}+\gamma+1\right)\right\}. \quad (31)$$

The second term on the right-hand side implies a reduction of the drag $O(k, k|\ln k|)$ owing to the slip effect in the edge regions of the plate.

On the other hand, if the perturbation due to the slip in the central part of the plate is considered, it may be anticipated that the perturbation makes a contribution to the drag of the same order. Therefore it becomes necessary to take the main solution to the next approximation. An iteration procedure starting from (28), however, encounters difficulties because of the singularities of (28) at the edges of the plate. We therefore start from the uniformly valid solution $f_u(x)$ given by (30) and put

$$f(x) = f_u(x) + \hat{f}(x) \quad (\hat{f}(x) \ll f_u(x)). \quad (32)$$

Substituting (32) into (8) and retaining the terms $O(k)$, we obtain an integral equation for $\hat{f}(x)$ as follows (see appendix B):

$$\frac{1}{2\pi}\int_0^1\left(\ln|x-\xi| - \ln\frac{4}{R}+\gamma-1\right)\hat{f}(\xi)\,d\xi$$
$$\doteqdot \frac{2k}{\ln(16/R)-\gamma+1}\left[\left\{\frac{1}{[x(1-x)]^{\frac{1}{2}}} - \frac{1}{x^{\frac{1}{2}}} - \frac{1}{(1-x)^{\frac{1}{2}}}\right\} + \frac{\ln(2/k)+\gamma+1}{\pi^2}\left\{\ln(1-x) + \frac{1}{x^{\frac{1}{2}}}\ln\frac{1+x^{\frac{1}{2}}}{1-x^{\frac{1}{2}}} + \ln x + \frac{1}{(1-x)^{\frac{1}{2}}}\ln\frac{1+(1-x)^{\frac{1}{2}}}{1-(1-x)^{\frac{1}{2}}}\right\} - \frac{4}{\pi^2}\int_0^1\left\{\frac{1}{1-x\eta^2} + \frac{1}{1-(1-x)\eta^2}\right\}\ln\eta\,d\eta - \frac{2}{\pi^2}\left(\ln\frac{4}{R}-\gamma+1\right)\left(\ln\frac{2}{k}+\gamma+1\right)\right] \quad (0 < x < 1). \quad (33)$$

It is possible to calculate the integral

$$\int_0^1 \hat{f}(x)\,dx$$

directly from this equation without solving $\hat{f}(x)$ itself. (If necessary the solution can be obtained by the Carleman method; see Carrier, Krook & Pearson 1966, p. 428.) We multiply (33) by $1/[x(1-x)]^{\frac{1}{2}}$ and then integrate with respect to x from zero to unity. The result is

$$\int_0^1 \hat{f}(x)\,dx \doteqdot \frac{8k}{\pi}\frac{\ln(2/k)+\gamma+1}{\ln(16/R)-\gamma+1}\left\{1-\frac{\pi}{\ln(16/R)-\gamma+1}\right\}. \tag{34}$$

Thus the small perturbation over the main portion of the plate makes a contribution to the drag comparable to that arising from the narrow edge regions.

Taking the conventional definition of the drag coefficient†

$$C_D = \frac{4}{R}\int_0^1 \frac{\partial u}{\partial y}\bigg|_{y=0} dx = \frac{4}{R}\int_0^1 f(x)\,dx \tag{35}$$

and substituting (32) together with (31) and (34), we get

$$C_D \doteqdot \frac{8\pi}{R\{\ln(16/R)-\gamma+1\}}\left\{1-\frac{4k}{\pi}\frac{\ln(2/k)+\gamma+1}{\ln(16/R)-\gamma+1}\right\}. \tag{36}$$

Here the slip coefficient k is proportional to the Knudsen number K for a slightly rarefied gas. Also, there is the well-known relation $KR = \lambda S$, where $S = U/c_m$ is the speed ratio, c_m being the most probable molecular speed, and $\lambda = \frac{16}{5}\pi^{-\frac{1}{2}}$ for spherical molecules. Therefore k is proportional to S/R and so C_D as given by (36) may be regarded as a function of R and S. Thus in figure 2, C_D is plotted *vs.* R, by taking simply $k = K = \lambda S/R$. In the same figure C_D's for free molecular flow ($K\to\infty$) from kinetic theory are also included for comparison. Tamada & Inoue (1976) studied the slip flow past an elliptic cylinder by a (regular) perturbation method. They could not discuss, however, the case of a flat plate as a limit of their results. The appearance of the $k \ln k$ term above indicates the cause of their difficulties.

Finally, we present approximate expressions for the velocity field around the leading edge of the plate. When the kernel approximation is made in the integral equation (8), the velocity field (4) near the plate becomes

$$u = \frac{1}{2\pi}\int_0^1\left[\tfrac{1}{2}\ln\{(\xi-x)^2+y^2\}-\ln\frac{4}{R}+\gamma-\frac{(\xi-x)^2}{(\xi-x)^2+y^2}\right]f(\xi)\,d\xi, \tag{37a}$$

$$v = \frac{1}{2\pi}\int_0^1 \frac{(\xi-x)\,y}{(\xi-x)^2+y^2}f(\xi)\,d\xi. \tag{37b}$$

Substituting into these the uniformly valid solution (30) and carrying out some reductions (see appendix C), we obtain the following results (where $r = (x^2+y^2)^{\frac{1}{2}}$, $\theta = \tan^{-1}(y/x)$ and $s = r/k$):

$$1+u \sim (kA/2\pi)\{2\pi+(s\ln s)\cos\theta+2s(\pi-\theta)\sin\theta+(\gamma-\ln 2-2)\,s\cos\theta\}, \tag{38a}$$

$$v \sim (kA/2\pi)(\ln s+\gamma-\ln 2-1)\,s\sin\theta \tag{38b}$$

† $C_D = D/(\frac{1}{2}\rho U^2 l)$, where D is the drag force per unit span of the plate and ρ is the density of the fluid.

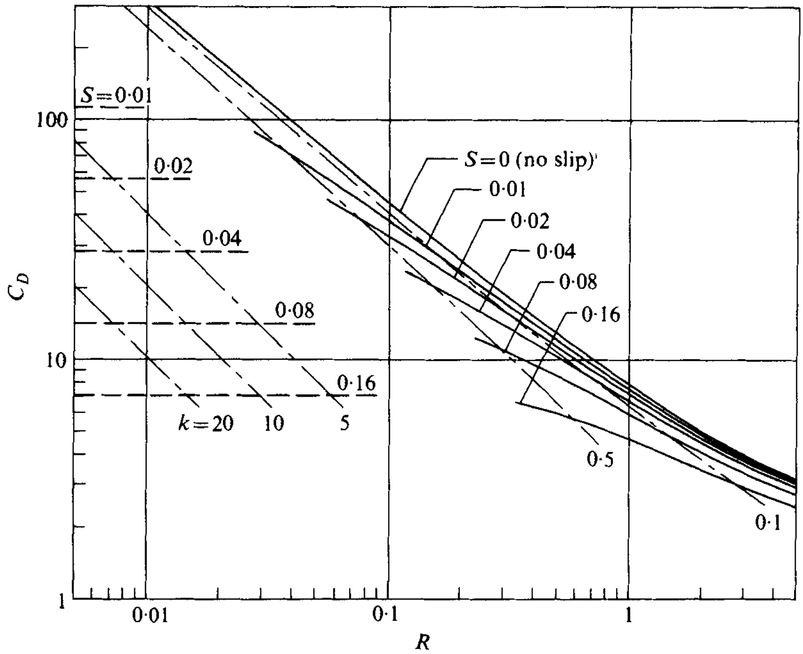

FIGURE 2. Variation of drag coefficient with Reynolds number. ——, equation (36); ---, free molecular flow ($C_D = 2/\pi^{\frac{1}{2}}S$).

for $s \ll 1$ $(r \ll k)$ and

$$1+u \sim [kA/2(2\pi)^{\frac{1}{2}}]\,[s^{\frac{1}{2}}(\sin\tfrac{3}{2}\theta+5\sin\tfrac{1}{2}\theta)+\pi^{-1}s^{-\frac{1}{2}}\{(\sin\tfrac{5}{2}\theta+3\sin\tfrac{1}{2}\theta)\ln s + (\cos\tfrac{5}{2}\theta+3\cos\tfrac{1}{2}\theta)\,(\pi-\theta)+(\gamma+\ln 2-1)\sin\tfrac{5}{2}\theta + (3\gamma+3\ln 2+5)\sin\tfrac{1}{2}\theta\}], \quad (39a)$$

$$v \sim [kA/2(2\pi)^{\frac{1}{2}}]\,[s^{\frac{1}{2}}(-\cos\tfrac{3}{2}\theta+\cos\tfrac{1}{2}\theta)+\pi^{-1}s^{-\frac{1}{2}}\{(-\cos\tfrac{5}{2}\theta+\cos\tfrac{1}{2}\theta)\ln s + (\sin\tfrac{5}{2}\theta-\sin\tfrac{1}{2}\theta)\,(\pi-\theta)-(\gamma+\ln 2-1)\cos\tfrac{5}{2}\theta+(\gamma+\ln 2-1)\cos\tfrac{1}{2}\theta\}] \quad (39\,b)$$

for $k^{-1} \gg s \gg 1$ $(1 \gg r \gg k)$. It is seen from (38) that the slip velocity at the leading edge takes the value $kA = (2\pi k)^{\frac{1}{2}}/\{\ln(16/R)-\gamma+1\} = O(k^{\frac{1}{2}})$ and the change in velocity near the edge is logarithmically singular. In (39), for large s, the first terms ($O(s^{\frac{1}{2}})$) correspond to no-slip flow while the second terms represent the additional velocity due to slippage, which diminishes as s increases. Here the presence of the terms proportional to $s^{-\frac{1}{2}}\ln s$ reflects again the singular character of the perturbation field.

Appendix A. Integration of $f_u(x)$

Integrating (30) from zero to unity, we have

$$\int_0^1 f_u(x)\,dx = \frac{2}{\ln(16/R)-\gamma+1}\left[\pi-4+\left(\frac{2k}{\pi}\right)^{\frac{1}{2}}\int_0^\infty p^{-\frac{3}{2}}(p+\tfrac{1}{2})^{-\frac{1}{2}} \times\left\{\cos\left(-\frac{1}{2\pi}P(p)+\frac{\pi}{4}\right)-\cos\left(\frac{1}{k}p-\frac{1}{2\pi}P(p)+\frac{\pi}{4}\right)\right\}dp\right]. \quad (\text{A }1)$$

In order to estimate the integral on the right-hand side, it is convenient to express it as the sum of three integrals I_1, I_2 and I_3, where

$$I_1 = \frac{1}{2}\int_{-\infty}^{\infty} p^{-\frac{3}{2}}\left[(p+\tfrac{1}{2})^{-\frac{1}{2}}\exp\left\{-\frac{i}{2\pi}P(p)+\frac{\pi}{4}i\right\}-2^{\frac{1}{2}}\exp\left(\frac{\pi}{4}i\right)\right]dp, \quad \text{(A 2a)}$$

$$I_2 = \int_0^{\infty} p^{-\frac{3}{2}}\left\{1-2^{\frac{1}{2}}\cos\left(\frac{1}{k}p+\frac{\pi}{4}\right)\right\}dp, \quad \text{(A 2b)}$$

$$I_3 = \frac{1}{2}\int_{-\infty}^{\infty} p^{-\frac{3}{2}}\left[2^{\frac{1}{2}}\exp\left(\frac{i}{k}p+\frac{\pi}{4}i\right)-(p+\tfrac{1}{2})^{-\frac{1}{2}}\exp\left\{\frac{i}{k}p-\frac{i}{2\pi}P(p)+\frac{\pi}{4}i\right\}\right]dp, \quad \text{(A 2c)}$$

with the path of integration deformed around the point $p = 0$ in the counter-clockwise sense. Then I_1 and I_2 become

$$I_1 = 0, \quad I_2 = 2(2\pi/k)^{\frac{1}{2}}, \quad \text{(A 3)}$$

and the asymptotic expansion of I_3 for small k is easily obtained in the form (cf. Carslaw & Jaeger 1947, p. 271)

$$I_3 \doteqdot -2\left(\frac{2k}{\pi}\right)^{\frac{1}{2}}\left(\ln\frac{2}{k}+\gamma+1\right). \quad \text{(A 4)}$$

Thus (A 1), together with (A 3) and (A 4), gives the formula (31).

Appendix B. Integral equation for refinement of main solution

When (30) and (32) are substituted into the integral equation (8) it takes the form

$$\frac{1}{2\pi}\int_0^1\left(\ln|x-\xi|-\ln\frac{4}{R}+\gamma-1\right)\hat{f}(\xi)\,d\xi$$
$$= k\{f_u(x)+\hat{f}(x)\}-\frac{1}{2\pi}\int_0^1 \ln|x-\xi|\{f_e(\xi)+f_e(1-\xi)\}\,d\xi$$
$$+\frac{1}{\pi\{\ln(16/R)-\gamma+1\}}\int_0^1 \ln|x-\xi|\left(\frac{1}{\xi^{\frac{1}{2}}}+\frac{1}{(1-\xi)^{\frac{1}{2}}}\right)d\xi$$
$$+\frac{1}{2\pi}\left(\ln\frac{4}{R}-\gamma+1\right)\int_0^1\left[f_u(\xi)-\frac{2}{\{\ln(16/R)-\gamma+1\}[\xi(1-\xi)]^{\frac{1}{2}}}\right]d\xi \quad (0 < x < 1). \quad \text{(B 1)}$$

We must estimate the terms on the right-hand side in the central part of the plate, where $x = O(1)$ and $1-x = O(1)$. We have

$$f_u(x) \doteqdot \frac{2}{\ln(16/R)-\gamma+1}\frac{1}{[x(1-x)]^{\frac{1}{2}}}. \quad \text{(B 2)}$$

Note that the edge solution $f_e(x)$ satisfies the relation

$$\frac{1}{2\pi}\int_0^{\infty}\ln\left|\frac{x-\xi}{1-\xi}\right|f_e(\xi)\,d\xi = k\{f_e(x)-f_e(1)\}, \quad \text{(B 3)}$$

as is seen from the integration of (11) with respect to x from unity to x. It follows from this that

$$\frac{1}{2\pi}\int_0^1 \ln|x-\xi|f_e(\xi)\,d\xi = k\{f_e(x)-f_e(1)\}-\frac{1}{2\pi}\int_1^{\infty}\ln\left(1-\frac{x}{\xi}\right)f_e(\xi)\,d\xi$$
$$+\frac{1}{2\pi}\int_1^{\infty}\ln\left(1-\frac{1}{\xi}\right)f_e(\xi)\,d\xi+\frac{1}{2\pi}\int_0^1\ln(1-\xi)f_e(\xi)\,d\xi. \quad \text{(B 4)}$$

Taking into account the asymptotic form (27) of $f_e(x)$ for $x \gg k$, we can express the right-hand side in the following approximate form:

$$\frac{1}{2\pi}\int_0^1 \ln|x-\xi| f_e(\xi)\,d\xi \doteqdot \frac{2}{\pi\{\ln(16/R)-\gamma+1\}}\left[\ln(1-x)+x^{\frac{1}{2}}\ln\frac{1+x^{\frac{1}{2}}}{1-x^{\frac{1}{2}}}-2\right]$$
$$-\frac{2k}{\pi^2\{\ln(16/R)-\gamma+1\}}\left[-\frac{\pi^2}{x^{\frac{1}{2}}}+\left(\ln\frac{2}{k}+\gamma+1\right)\right.$$
$$\left.\times\left\{\ln(1-x)+\frac{1}{x^{\frac{1}{2}}}\ln\frac{1+x^{\frac{1}{2}}}{1-x^{\frac{1}{2}}}\right\}-4\int_0^1\frac{\ln\eta}{1-x\eta^2}d\eta\right]. \quad \text{(B 5)}$$

We also get

$$\frac{1}{2\pi}\int_0^1 \ln|x-\xi| f_e(1-\xi)\,d\xi = \frac{1}{2\pi}\int_0^1 \ln|1-x-\xi| f_e(\xi)\,d\xi, \quad \text{(B 6)}$$

$$\frac{1}{\pi\{\ln(16/R)-\gamma+1\}}\int_0^1 \ln|x-\xi|\left(\frac{1}{\xi^{\frac{1}{2}}}+\frac{1}{(1-\xi)^{\frac{1}{2}}}\right)d\xi$$
$$= \frac{2}{\pi\{\ln(16/R)-\gamma+1\}}\left[\ln(1-x)+x^{\frac{1}{2}}\ln\frac{1+x^{\frac{1}{2}}}{1-x^{\frac{1}{2}}}+\ln x+(1-x)^{\frac{1}{2}}\ln\frac{1+(1-x)^{\frac{1}{2}}}{1-(1-x)^{\frac{1}{2}}}-4\right]. \quad \text{(B 7)}$$

On substituting these results together with (31) into (B 1) and retaining the terms $O(k)$, we obtain the integral equation (33) for $\hat{f}(x)$.

Appendix C. Velocity field in the vicinity of the edge

In order to get approximate expressions for the velocity components in the region where $(x^2+y^2)^{\frac{1}{2}} \ll k$, we rewrite (37) on integration by parts in the form

$$u = \frac{1}{2\pi}\int_0^1\left(\ln\xi-\ln\frac{4}{R}+\gamma-1\right)f(\xi)\,d\xi$$
$$+\frac{1}{2\pi}f(1)\left[\tfrac{1}{2}(1-x)\ln\{(1-x)^2+y^2\}+x+2|y|\tan^{-1}\frac{1-x}{|y|}\right]$$
$$+\frac{1}{2\pi}f(0)\left[\tfrac{1}{2}x\ln(x^2+y^2)-x+2|y|\tan^{-1}\frac{x}{|y|}\right]-\frac{1}{2\pi}\int_0^1(-x\ln\xi+\pi|y|)\frac{df}{d\xi}d\xi$$
$$-\frac{1}{2\pi}\int_0^1\left[\tfrac{1}{2}(\xi-x)\ln\{(\xi-x)^2+y^2\}-\xi\ln\xi+x+x\ln\xi+2|y|\tan^{-1}\frac{\xi-x}{|y|}-\pi|y|\right]\frac{df}{d\xi}d\xi, \quad \text{(C 1a)}$$

$$v = \frac{1}{4\pi}y[f(1)\ln\{(1-x)^2+y^2\}-f(0)\ln(x^2+y^2)]$$
$$-\frac{1}{2\pi}y\int_0^1\ln\xi\frac{df}{d\xi}d\xi-\frac{1}{4\pi}y\int_0^1[\ln\{(\xi-x)^2+y^2\}-2\ln\xi]\frac{df}{d\xi}d\xi. \quad \text{(C 1b)}$$

We estimate these formulae using the uniformly valid solution $f_u(x)$ for $f(x)$ to the first approximation. The last two terms in these equations are of higher order than the rest for small $(x^2+y^2)^{\frac{1}{2}}/k$, and can be neglected. Taking into consideration that the edge solution $f_e(x)$ satisfies (11) and that its asymptotic behaviour for small and large x/k is given by (26) and (27), we can derive the following results:

$$f_u(0) = f_u(1) \doteqdot A, \tag{C 2}$$

$$\frac{1}{2\pi}\int_0^1 \left(\ln\xi - \ln\frac{4}{R} + \gamma - 1\right) f_u(\xi)\, d\xi \doteqdot -1 + Ak, \tag{C 3}$$

$$\frac{1}{2\pi}\int_0^1 \ln\xi \frac{df_u}{d\xi} d\xi \doteqdot \frac{A}{2\pi}\left(\ln\frac{1}{2k} + \gamma - 1\right). \tag{C 4}$$

Substitution of these results into (C 1) leads to the approximate formula (38). By integrating (37) by parts in a different way and making some reductions, we can obtain the formula (39) valid for $1 \gg (x^2+y^2)^{\frac{1}{2}} \gg k$.

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